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Small torsion generating sets for hyperelliptic mapping class groups

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Abstract

We prove that both the hyperelliptic mapping class group and the extended hyperelliptic mapping class group are generated by two torsion elements. We also compute the index of the subgroup of the hyperelliptic mapping class group which is generated by involutions and we prove that the extended hyperelliptic mapping class group is generated by three orientation reversing involutions.

Keywords: Mapping class group; Hyperelliptic mapping class group; Symmetries; Torsion generators; Involutions

1. Introduction

Let S_g be a closed orientable surface of genus $g \geqslant 2$. Denote by $\mathcal{M}_g^{\pm}$ the *extended mapping class group*, i.e., the group of isotopy classes of homeomorphisms of S_g. By $\mathcal{M}_g$ we denote the *mapping class group*, i.e., the subgroup of $\mathcal{M}_g^{\pm}$ consisting of orientation preserving maps. We will make no distinction between a map and its isotopy class, so in particular by the order of a homeomorphism $h: S_g \to S_g$ we mean the order of its class in $\mathcal{M}_g^{\pm}$.

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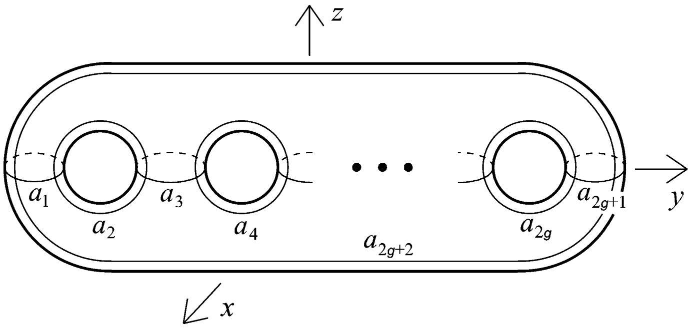

Fig. 1. Surface S_g embedded in $\mathbb{R}^3$.

Suppose that S_g is embedded in $\mathbb{R}^3$ as shown in Fig. 1, in such a way that it is invariant under reflections across xy, yz, xz planes. Let $\varrho : S_g \to S_g$ be a *hyperelliptic involution*, i.e., the half turn about the y-axis.

The *hyperelliptic mapping class group* $\mathcal{M}_g^h$ is defined to be the centraliser of ϱ in $\mathcal{M}_g$. In a similar way, we define the *extended hyperelliptic mapping class group* $\mathcal{M}_g^{h\pm}$ to be the centraliser of ϱ in $\mathcal{M}_g^{\pm}$. For $g = 2$ it is known that $\mathcal{M}_2 = \mathcal{M}_2^h$ and $\mathcal{M}_2^{\pm} = \mathcal{M}_2^{h\pm}$.

The problem of finding certain generating sets for groups $\mathcal{M}_g$ and $\mathcal{M}_g^{\pm}$ has been studied extensively—see [2–8,10,11] and references there. In particular it is known that the group $\mathcal{M}_g$ could be generated by 2 torsion elements [6], and by small number of involutions [5]. The problem of finding the minimal generating set consisting of involutions is still open.

Similar results hold for the extended mapping class group, namely the group $\mathcal{M}_g^{\pm}$ is generated by 2 elements [6], and by 3 symmetries (i.e., orientation reversing involutions) [10]. The question if it is possible to generate $\mathcal{M}_g^{\pm}$ by two torsion elements is open.

The purpose of this paper is to give a full answer to analogous questions in the case of the hyperelliptic mapping class group and the extended hyperelliptic mapping class group, i.e., we will show that both groups are generated by two torsion elements and $\mathcal{M}_g^{h\pm}$ is generated by three symmetries. It will be also observed that the subgroup I_g of $\mathcal{M}_g^h$ which is generated by involutions is a proper subgroup and we will compute the index $[\mathcal{M}_g^h : I_g]$.

The importance of torsion elements in the group $\mathcal{M}_g$ ($\mathcal{M}_g^{\pm}$) follows from the fact that any such element could be realized as an analytic (dianalytic) automorphism of some Riemann surface. Similarly torsion elements in $\mathcal{M}_g^h$ ($\mathcal{M}_g^{h\pm}$) correspond to analytic (dianalytic) automorphisms of hyperelliptic Riemann surfaces, i.e., complex algebraic curves with affine part defined by an equation $y^2 = f(x)$, where f is a polynomial with distinct roots.

Notice that since $\mathcal{M}_2 = \mathcal{M}_2^h$ and $\mathcal{M}_2^{\pm} = \mathcal{M}_2^{h\pm}$, our results imply that $\mathcal{M}_2$ and $\mathcal{M}_2^{\pm}$ are generated by two torsion elements.

2. Preliminaries

Let $A_1, \ldots, A_{2g+2}$ be the right Dehn twists along the curves $a_1, \ldots, a_{2g+2}$ indicated in Fig. 1. Denote also $B = A_1A_2 \cdots A_{2g+1}$ and $\overline{B} = A_{2g+1} \cdots A_2A_1$. It is known [1] that

$$\varrho = B\overline{B} = A_1A_2 \cdots A_{2g+1}A_{2g+1} \cdots A_2A_1$$

and $\mathcal{M}_g^h$ admits the presentation:
generators: $A_1, \ldots, A_{2g+2}, B, \varrho$
defining relations:

$$\varrho = A_1A_2 \cdots A_{2g+1}A_{2g+1} \cdots A_2A_1 \tag{1}$$
$$B = A_1A_2 \cdots A_{2g+1} \tag{2}$$
$$A_j = BA_iB^{-1} \quad j \equiv i+1 \mod 2g+2 \tag{3}$$
$$A_iA_j = A_jA_i, \quad 2 \leqslant |i-j| \leqslant 2g \tag{4}$$
$$A_iA_jA_i = A_jA_iA_j \quad j \equiv i+1 \mod 2g+2 \tag{5}$$
$$B^{2g+2} = 1 \tag{6}$$
$$\varrho^2 = 1 \tag{7}$$
$$\varrho A_i = A_i\varrho \tag{8}$$

The above presentation follows from the presentation in [1] by adding generators A_{2g+2}, B, ϱ and some superfluous relations. The relation (3) follows from easily verified observation that

$$B(a_i) = a_j \quad \text{for } j \equiv i+1 \mod 2g+2$$

Combining this with the relation (6) we have that B has order $2g+2$. Observe also, that for $k \in \mathbb{Z}$

$$B = B^kBB^{-k} = B^kA_1A_2 \cdots A_{2g+1}B^{-k} = A_{1+k}A_{2+k} \cdots A_{2g+1+k}$$

where subscripts should be reduced modulo $2g+2$.

Let us also point out that the relation (3) implies that $\langle B, A_i\rangle = \mathcal{M}_g^h$ for every $1 \leqslant i \leqslant 2g+2$.

3. Minimal torsion generating sets for $\mathcal{M}_g^h$ and $\mathcal{M}_g^{h\pm}$

Theorem 1. *For every $g \geqslant 2$ the group $\mathcal{M}_g^h$ is generated by two elements of order $2g+2$ and $4g+2$ respectively.*

Proof. Let $M = A_2A_3 \cdots A_{2g+1}$. It is well known that M has order $4g+2$ (cf. [2,10]). Since $A_1 = BM^{-1}$, we have

$$\langle B, M\rangle = \langle B, A_1\rangle = \mathcal{M}_g^h \qquad \square$$

Let σ be the reflection across the yz-plane (Fig. 1). Since $\sigma(a_i)=a_i^{\pm 1}$, $1\leqslant i\leqslant 2g+2$ and σ reverses orientation, we have

$$\sigma A_i\sigma = A_i^{-1}\quad \text{for } 1\leqslant i\leqslant 2g+2 \tag{9}$$

Therefore

$$\sigma B\sigma = \overline{B}^{-1}=\varrho B \tag{10}$$

Lemma 2. *The order of the element $\beta=\sigma B$ is finite and equal to $2g+2$ for g odd and $4g+4$ for g even.*

Proof. Since $\beta^2=\sigma B\sigma B=\varrho B^2$, we have $\beta^{2g+2}=\varrho^{g+1}$, which completes the proof. □

Lemma 3. *The order of the element $N=\sigma A_{2g+1}^{-1}A_1A_2A_1^{-1}BA_{2g+1}^{-1}$ is finite and equal to $2g$ for g odd and $4g$ for g even.*

Proof. Using relations (3)–(10), we compute

$$\begin{aligned}
N^2 &= \left(\sigma A_{2g+1}^{-1}A_1A_2A_1^{-1}BA_{2g+1}^{-1}\right)\left(\sigma A_{2g+1}^{-1}A_1A_2A_1^{-1}BA_{2g+1}^{-1}\right)\\
&= A_{2g+1}A_1^{-1}A_2^{-1}A_1\varrho BA_1A_2A_1^{-1}BA_{2g+1}^{-1}\\
&= A_{2g+1}\left(A_1^{-1}A_2^{-1}A_1\right)A_2A_3A_2^{-1}B^2A_{2g+1}^{-1}\varrho\\
&= A_{2g+1}\left(A_2A_1^{-1}A_2^{-1}\right)A_2A_3A_2^{-1}B^2A_{2g+1}^{-1}\varrho\\
&= A_{2g+1}(A_2A_3)\left(A_1^{-1}A_2^{-1}\right)B^2A_{2g+1}^{-1}\varrho
\end{aligned} \tag{11}$$

Similar computations show that

$$\begin{aligned}
N^{2g} &= A_{2g+1}(A_2\cdots A_{2g}A_{2g+1})\left(A_1^{-1}\cdots A_{2g-1}^{-1}A_{2g}^{-1}\right)B^{2g}A_{2g+1}^{-1}\varrho^g\\
&= A_{2g+1}(A_2\cdots A_{2g}A_{2g+1}A_{2g+2})\left(A_{2g+2}^{-1}A_1^{-1}\cdots A_{2g-1}^{-1}A_{2g}^{-1}\right)B^{2g}A_{2g+1}^{-1}\varrho^g\\
&= A_{2g+1}B\varrho BB^{2g}A_{2g+1}^{-1}\varrho^g=\varrho^{g+1}
\end{aligned}$$

This completes the proof. □

Let $G=\langle\beta,N\rangle$. We will show that $G=\mathcal{M}_g^{h\pm}$, but first we need the following lemma.

Lemma 4. *If $g\geqslant 3$ then $A_{2g+1}A_1^{-1}\in G$.*

Proof. Suppose first that $g\geqslant 4$. Using (11) we have

$$\begin{aligned}
&N^{-2}\left(\beta^4N^2\beta^{-4}\right)\left(\beta^{-2}N^2\beta^2\right)\left(\beta^2N^{-2}\beta^{-2}\right)\\
&\quad= \left(A_{2g+1}B^{-2}A_2A_1A_3^{-1}A_2^{-1}A_{2g+1}^{-1}\right)\varrho\left(A_3A_6A_7A_5^{-1}A_6^{-1}B^2A_3^{-1}\right)\varrho\\
&\qquad\times\left(A_{2g-1}A_{2g+2}A_1A_{2g+1}^{-1}A_{2g+2}^{-1}B^2A_{2g-1}^{-1}\right)\varrho\left(A_1B^{-2}A_4A_3A_5^{-1}A_4^{-1}A_1^{-1}\right)\varrho\\
&\quad= A_{2g+1}A_{2g+2}A_{2g+1}A_1^{-1}A_{2g+2}^{-1}A_{2g-1}^{-1}A_1\left(A_4A_5A_3^{-1}A_4^{-1}A_3^{-1}\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \bigl(A_{2g-1}A_{2g+2}A_1A_{2g+1}^{-1}A_{2g+2}^{-1}A_{2g+1}^{-1}\bigr)A_3A_4A_3A_5^{-1}A_4^{-1}A_1^{-1}\\
&= A_{2g+1}A_{2g+2}A_{2g+1}A_1^{-1}A_{2g+2}^{-1}A_{2g-1}^{-1}A_1\bigl(A_{2g-1}A_{2g+2}A_1A_{2g+1}^{-1}A_{2g+2}^{-1}A_{2g+1}^{-1}\bigr)\\
&\quad\times \bigl(A_4A_5A_3^{-1}A_4^{-1}A_3^{-1}\bigr)A_3A_4A_3A_5^{-1}A_4^{-1}A_1^{-1}\\
&= A_{2g+1}A_{2g+2}A_{2g+1}A_1^{-1}A_{2g+2}^{-1}(A_1A_{2g+2}A_1)\bigl(A_{2g+1}^{-1}A_{2g+2}^{-1}A_{2g+1}^{-1}\bigr)A_1^{-1}\\
&= A_{2g+1}A_{2g+2}A_{2g+1}A_1^{-1}A_{2g+2}^{-1}(A_{2g+2}A_1A_{2g+2})\bigl(A_{2g+2}^{-1}A_{2g+1}^{-1}A_{2g+2}^{-1}\bigr)A_1^{-1}\\
&= A_{2g+1}A_1^{-1}
\end{aligned}
$$

If $g=3$ similar, but rather long computations[2] show that

$$\beta^{-4}N^{-2}\beta N^{-2}\beta^{-1}N^{-1}\beta^2N^2\beta^{-4}N\beta N^{-3}\beta^4N^{-1}\beta = A_7A_1^{-1} \qquad \square$$

Theorem 5. *For every $g \geqslant 2$ the group $\mathcal{M}_g^{h\pm}$ is generated by two elements of finite order.*

Proof. Let β and N be elements defined above. Since β satisfies the relation

$$A_j^{-1} = \beta A_i\beta^{-1} \quad j \equiv i+1 \mod 2g+2$$

to prove the theorem it is enough to show that $A_i \in G = \langle \beta, N\rangle$ for some $1 \leqslant i \leqslant 2g+2$.

If $g \geqslant 3$ then from Lemma 4 and the above relation follows that

$$A_{2g+1}A_1^{-1},\ A_{2g}A_{2g+2}^{-1},\ A_{2g-1}A_{2g+1}^{-1} \in G$$

Therefore

$$
\begin{aligned}
G &\ni \bigl(A_{2g-1}A_{2g+1}^{-1}\bigr)\bigl(A_{2g+1}A_1^{-1}\bigr)\bigl(A_{2g+2}^{-1}A_{2g}\bigr)\beta^{-1}N\bigl(A_{2g+1}A_{2g-1}^{-1}\bigr)\\
&= A_{2g-1}A_1^{-1}A_{2g+2}^{-1}A_{2g}B^{-1}A_{2g+1}^{-1}A_1A_2A_1^{-1}BA_{2g+1}^{-1}A_{2g+1}A_{2g-1}^{-1}\\
&= A_{2g-1}A_1^{-1}A_{2g+2}^{-1}A_{2g}A_{2g}^{-1}A_{2g+2}A_1A_{2g+2}^{-1}A_{2g-1}^{-1}\\
&= A_{2g-1}A_{2g+2}^{-1}A_{2g-1}^{-1} = A_{2g+2}^{-1}
\end{aligned}
$$

If $g=2$ then one could verify that

$$N^{-1}\beta N^2\beta N\beta N^{-1}\beta^{-1}N^{-1}\beta N^2\beta N\beta = A_3^{-1} \qquad \square$$

4. Involutions as generators for $\mathcal{M}_g^h$ and $\mathcal{M}_g^{h\pm}$

Observe that from a presentation for the group $\mathcal{M}_g^h$ we obtain

$$H_1\bigl(\mathcal{M}_g^h, \mathbb{Z}\bigr) = \begin{cases} \mathbb{Z}_{4g+2} & \text{for } g \text{ even}\\ \mathbb{Z}_{8g+4} & \text{for } g \text{ odd}\end{cases}$$

Since neither $\mathbb{Z}_{4g+2}$ nor $\mathbb{Z}_{8g+4}$ is generated by involutions the same conclusion holds for $\mathcal{M}_g^h$.

[2] There are available at the URL: http://www.math.univ.gda.pl/~trojkat/comgenhi.pdf.

Denote by $I_g \leqslant \mathcal{M}_g^h$ the subgroup generated by involutions. Clearly this is a normal subgroup of $\mathcal{M}_g^h$. Our next goal is to describe the quotient $\mathcal{M}_g^h/I_g$. To achieve it we will follow similar lines to [8].

Let S be the half turn S about the z-axis (Fig. 1).

Lemma 6. *The number of conjugacy classes of involutions in $\mathcal{M}_g^h$ is equal to 2 for g even and 3 for g odd. These classes are represented by ϱ, S and ϱ, S, ϱS respectively.*

Proof. Since ϱ is central, its conjugacy class consists of one element. Therefore we could restrict ourselves to conjugacy classes of involutions different from ϱ.

Let $R \in \mathcal{M}_g^h$ be an involution and $H = \langle R, \varrho\rangle$. In particular H has order 4 and contains ϱ. From Theorem 4 in [9] follows that there are exactly 2 conjugacy classes of subgroups of $\mathcal{M}_g^h$ having these two properties. To identify these conjugacy classes, it is enough to find two nonconjugate examples of such subgroups. The first one is the dihedral group $\langle S, \varrho\rangle$. An example of a subgroup in the second conjugacy class follows from easily verified fact that

$$\left(A_1 A_{2g+2}^{-1} B\right)^{2g} = \varrho$$

In particular the group $\langle (A_1 A_{2g+2}^{-1} B)^g\rangle$ is a cyclic group of order 4 representing the second conjugacy class. Therefore H, as a dihedral group, is conjugate to $\langle S, \varrho\rangle$, hence R is conjugate to either S or ϱS. To complete the proof it is enough to show that S and ϱS are conjugate if and only if g is even.

First suppose that g is even. Let t be a circle fixed by S, and

$$\theta = (A_{g+2} A_{g+3} \cdots A_{2g} A_{2g+1})^{g+1}$$

From geometric point of view, θ is a half-twist about t, i.e., it is a half-turn of the right half of T_g, in particular $\theta^2 = T$—twist along t. From this geometric interpretation it is clear that

$$\theta\left(S\theta^{-1}S^{-1}\right) = \varrho$$

Hence $\theta S \theta^{-1} = \varrho S$.

On the other hand, if g is odd, then automorphisms induced by S and ϱS on $H_1(T_g, \mathbb{Z})$ have different eigenvalues, so S and ϱS cannot be conjugate in $\mathcal{M}_g^h$. $\square$

Lemma 7. *The quotient $\mathcal{M}_g^h/I_g$ is cyclic.*

Proof. Since $S(a_1) = a_{2g+1}$ we have

$$A_1 A_{2g+1}^{-1} = A_1\left(S A_1^{-1} S^{-1}\right) = \left(A_1 S A_1^{-1}\right) S^{-1} \in I_g$$

Now observe that for any $3 \leqslant i \leqslant 2g$, we can construct an element $F \in \mathcal{M}_g^h$ such that $F(a_1) = a_1$ and $F(a_i) = a_{2g+1}$. In fact, if we define $F_j = A_j A_{j+1}$ for $3 \leqslant j \leqslant 2g$ then $F_j(a_1) = a_1$, $F_j(a_j) = a_{j+1}$ so we could take $F = F_{2g} F_{2g-1} \cdots F_i$. Hence

$$A_1 A_i^{-1} = F^{-1}\left(A_1 A_{2g+1}^{-1}\right) F \in I_g \quad \text{for } 3 \leqslant i \leqslant 2g$$

Finally we have

$$A_{2g+1}A_2^{-1} = S\left(A_1A_{2g}^{-1}\right)S^{-1} \in I_g$$

Therefore all twists $A_1, \ldots, A_{2g+1}$ are equal modulo I_g. Since they generate $\mathcal{M}_g^h$ this implies that $\mathcal{M}_g^h/I_g$ is cyclic. $\square$

Theorem 8. *The index* $[\mathcal{M}_g^h : I_g]$ *of the subgroup generated by involutions is equal to* $2g+1$ *for* g *even and* $4g+2$ *for* g *odd.*

Proof. Let $\pi : \mathcal{M}_g^h \to H_1(\mathcal{M}_g^h, \mathbb{Z})$ be the canonical projection. By Lemma 7, $[\mathcal{M}_g^h, \mathcal{M}_g^h] \leqslant I_g$, so

$$\left[\mathcal{M}_g^h : I_g\right] = \left[H_1\left(\mathcal{M}_g^h, \mathbb{Z}\right) : \pi(I_g)\right] \tag{12}$$

From the presentation for $\mathcal{M}_g^h$, we have that for any $1 \leqslant i \leqslant 2g+2$ the group $H_1(\mathcal{M}_g^h, \mathbb{Z})$ is generated by $\pi(A_i)$ and

$$H_1\left(\mathcal{M}_g^h, \mathbb{Z}\right) = \begin{cases} \mathbb{Z}_{4g+2} & \text{for } g \text{ even} \\ \mathbb{Z}_{8g+4} & \text{for } g \text{ odd} \end{cases}$$

Now observe that since B has order $2g+2$, B^{g+1} is an involution. It is not central in $\mathcal{M}_g^h$, so it is conjugate to S or to ϱS. Therefore, using relations (1), (2) and Lemma 6 we obtain

$$\begin{aligned} \pi(I_g) &= \left\langle \pi(\varrho), \pi(S), \pi(\varrho S)\right\rangle \\ &= \left\langle \pi(\varrho), \pi\left(B^{g+1}\right)\right\rangle \\ &= \left\langle 2(2g+1), (g+1)(2g+1)\right\rangle \end{aligned}$$

Together with (12) this gives us the desired result. $\square$

Theorem 9. *The group* $\mathcal{M}_g^{h\pm}$ *is generated by three symmetries.*

Proof. Let τ be the reflection across the xz-plane (Fig. 1) and $\varepsilon_1 = \tau B$, $\varepsilon_2 = \tau A_{g+1}$. Since $\tau B \tau = B^{-1}$ and $\tau A_{g+1} \tau = A_{g+1}^{-1}$, both ε_1 and ε_2 are symmetries. Moreover $A_{g+1}, B \in \langle \tau, \varepsilon_1, \varepsilon_2\rangle$, so $\langle \tau, \varepsilon_1, \varepsilon_2\rangle = \mathcal{M}_g^{h\pm}$. $\square$

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